JOURNAL OF APPROXIMATION THEORY 5, 123–136 (1972)

Representation of Functions by Generalized Lidstone Series

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Communicated by E. W. Cheney

Received March 4, 1970

DEDICATED TO PROFESSOR J. L. WALSH ON THE OCCASION OF HIS 75TH BIRTHDAY

1. INTRODUCTION

A Lidstone series provides a two-point expansion of a given function $f(z)$ in the form

$$f(z) = f(1)\,\Lambda_0(z) + f(0)\,\Lambda_0(1-z) + f''(1)\,\Lambda_1(z) + f''(0)\,\Lambda_1(1-z) + \cdots, \tag{1.1}$$

where $\Lambda_n(z)$ is a polynomial of degree $2n+1$ defined by the generating function

$$\frac{\sinh zt}{\sinh t} = \sum_{n=0}^{\infty} t^{2n}\Lambda_n(z). \tag{1.2}$$

Lidstone series have received much attention by H. Poritsky, I. J. Schoenberg, D. V. Widder, R. P. Boas, Jr. and others. For relevant literature on the subject, see Boas [1].

A function $f(x)$, having derivatives of all orders on an interval $[a, b]$, with the additional properties $(-1)^n f^{(2n)}(x) \geqslant 0$ $(a \leqslant x \leqslant b;\ n = 0, 1,\ldots)$, is said to be *completely convex* on that interval. Widder [5, 6] studied the relationship between functions having a Lidstone series representation and the class of completely convex functions. He showed that a function $f(x)$ has an absolutely convergent Lidstone series representation if and only if it can be written as the difference of two minimal completely convex functions [6]. Boas [1] pointed out the difficulty in applying Widder's necessary and sufficient condition to an arbitrary function $f(x)$. He then gave simple necessary conditions and sufficient conditions for representation of functions by Lidstone series.

Recently, Leeming and Sharma [3] have introduced an extension of the

Lidstone series (called p-Lidstone or (p, L) series), and of completely convex functions (called completely W_p-convex functions), which led to a generalization of Widder's result [3, Theorem IV].

It is the object of this paper to give a generalization of the results of Boas [1] using the terminology and results given in [3].

In the remainder of Section 1 we give the notation and basic definitions used throughout the paper. In Section 2, we state and prove two theorems giving necessary conditions for representations of functions by (p, L) series. Section 3 contains four theorems which give sufficient conditions for representation of functions by (p, L) series.

Set

$$N_{p,j}(t) = \sum_{n=0}^{\infty} \frac{t^{pn+j}}{(pn+j)!}; \qquad M_{p,j}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{pn+j}}{(pn+j)!} \tag{1.3}$$

$(j = 0, 1, \dots, p-1; p = 2, 3, \dots)$.

We denote the positive zeros of $M_{p,p-1}(t)$ by

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots. \tag{1.4}$$

DEFINITION 1.1: The (formal) representation of a function $f \in C^\infty [0, 1]$

$$f(z) = \sum_{n=0}^{\infty} \left[f^{(pn)}(1)\, C_{pn}(z) + \sum_{j=0}^{\infty} f^{(pn+j)}(0)\, A_{pn+j}(z) \right] \tag{1.5}$$

where $\{C_{pn}(z)\}_{n=0}^{\infty}$ and $\{A_{pn+j}(z)\}_{n=0}^{\infty}$ $(j = 0, 1, \dots, p-2)$ are defined by the generating functions

$$\sum_{n=0}^{\infty} t^{pn} C_{pn}(z) = \frac{N_{p,p-1}(zt)}{N_{p,p-1}(t)} \tag{1.6}$$

$$\sum_{n=0}^{\infty} t^{pn+j} A_{pn+j}(z) = N_{p,j}(zt) - \frac{N_{p,j}(t)\, N_{p,p-1}(zt)}{N_{p,p-1}(t)} \tag{1.7}$$

$(j = 0, 1, \dots, p-2)$ and $N_{P,j}(t)$ is given by (1.3) is called the *p-Lidstone* (or (p, L)) *series* of $f(z)$.

DEFINITION 1.2. A real valued function f defined on $[a, b]$ is said to be *completely* W_p*-convex* if

(i) $f \in C^\infty[a, b]$

(ii) $(-1)^k f^{(pk)}(x) \geqslant 0 \qquad (a \leqslant x \leqslant b;\ k = 0, 1, \dots)$

(iii) $(-1)^k f^{(pk+j)}(a) \geqslant 0 \qquad (j = 1, 2, \dots, p-2;\ k = 0, 1, \dots)$.

Remark. If we set $p = 2$ in (1.5) we have the Lidstone series expansion (1.1). Further, setting $p = 2$ in Definition 1.2 we have the class of completely convex functions.

2. Some Necessary Conditions for Representation of Functions by (p, L) Series

We shall show that results similar to those of Boas for the Lidstone series [1] hold for representation of a function by a (p, L) series. In particular we prove,

THEOREM 2.1. *If the (p, L) series of $f(z)$ converges absolutely to $f(z)$ then*

$$f(z) = 0(e^{|z|\lambda_1}) \qquad (|z| \to \infty) \tag{2.1}$$

where λ_1 is given by (1.4).

THEOREM 2.2. *No condition of the form*

$$f(z) = 0\{\phi(|z|)\, e^{|z|\lambda_1}\} \qquad (|z| \to \infty) \tag{2.2}$$

where $\phi(r) \to 0$ as $r \to \infty$ and ϕ is independent of f is necessary for the absolute convergence of the (p, L) series to $f(z)$.

In order to prove Theorem 2.1 we require some preliminary results. Define

$$(p-1)!\,K_1(x,t) = \begin{cases} (x-t)^{p-1} - (1-t)^{p-1}\,x^{p-1}, & (0 \leqslant t < x \leqslant 1) \\ -(1-t)^{p-1},\, x^{p-1} & (0 \leqslant x \leqslant t \leqslant 1) \end{cases} \tag{2.3}$$

$K_1(x, t)$ is the Green's function for the differential system

$$\begin{cases} y^{(p)}(x) = \phi(x) \\ y(1) = 0;\ y(0) = y'(0) = \cdots = y^{(p-2)}(0) = 0, \end{cases} \tag{2.4}$$

where $\phi(x)$ is any function continuous on [0, 1] so that

$$y(x) = \int_0^1 K_1(x,t)\,\phi(t)\,dt. \tag{2.5}$$

If we denote the iterates of $K_1(x, t)$ by

$$K_n(x,t) = \int_0^1 K_1(x,u)\,K_{n-1}(u,t)\,du \qquad (n = 2, 3, \ldots), \tag{2.6}$$

then we have

LEMMA 2.1. *If* $f(x) \in C^\infty [0, 1]$, *then*

$$f(x) = \sum_{k=0}^{n-1} f^{(pk)}(1)\, C_{pk}(x) + \sum_{j=0}^{p-2} \sum_{k=0}^{n-1} f^{(pk+j)}(0)\, A_{pk+j}(x) + R_n(x, f), \qquad (2.7)$$

where

$$\{C_{pk}(x)\}_{k=0}^{n-1} \quad \textit{and} \quad \{A_{pk+j}(x)\}_{k=0}^{n-1} \qquad (j = 0, 1, ..., p-2)$$

are defined by (1.6) *and* (1.7) *and*

$$R_n(x, f) = \int_0^1 K_n(x, t) f^{(pn)}(t)\, dt \qquad (2.8)$$

with $K_n(x, t)$ *given by* (2.3) *and* (2.6).

LEMMA 2.2. *If* $f(x)$ *is completely* W_p*-convex on* $[0, 1]$ *then for any* $x_0 (0 \leqslant x_0 \leqslant 1)$,

$$f(x) \geqslant f(x_0)\, x^{p-1} \qquad (0 \leqslant x \leqslant x_0), \qquad (2.9)$$

$$f(x) \leqslant f(x_0)(1 - x^{p-1}) \qquad (x_0 \leqslant x \leqslant 1). \qquad (2.10)$$

Note. Throughout the paper we use B to denote suitable constants (not necessarily the same) which are independent of n and x unless otherwise stated.

LEMMA 2.3. (a) *For* $0 \leqslant x \leqslant 1$, $n = 1, 2, ...,$

$$0 \leqslant (-1)^n C_{pn}(x) \leqslant \frac{B}{\lambda_1^{pn}}, \qquad (2.11)$$

$$0 \leqslant (-1)^n A_{pn+j}(x) \leqslant \frac{B}{\lambda_1^{pn}} \qquad (n = 0, 1, ..., p-2). \qquad (2.12)$$

(b) *For any fixed* $x_0 (0 < x_0 < 1)$ *there is a constant* B *such that*

$$(-1)^n C_{pn}(x_0) \geqslant \frac{B}{\lambda_1^{pn}} \qquad (n = 1, 2, ...), \qquad (2.13)$$

$$(-1)^n A_{pn+j}(x_0) \geqslant \frac{B}{\lambda_1^{pn}} \qquad (j = 0, 1, ..., p-2;\ n = 1, 2, ...). \qquad (2.14)$$

The proofs of Lemmas 2.1–2.3 are given in [3].

LEMMA 2.4. *If $f(x)$ is completely W_p-convex on* [0, 1] *and*

$$M_n = \max_{0 \leqslant x \leqslant 1} |f^{(n)}(x)|,$$

then

$$(-1)^n f^{(pn)}(x) \geqslant M_{pn}(x^{p-1} - x^p), \qquad (0 \leqslant x \leqslant 1, \quad n = 0, 1, \ldots). \quad (2.15)$$

Proof. Suppose the maximum M_{pn} is attained at $x = x_0$. Then, by Lemma 2.1, we have for $0 \leqslant x \leqslant x_0$

$$(-1)^n f^{(pn)}(x) = (-1)^n \sum_{j=0}^{p-2} f^{(pn+j)}(0) \left[\left(\frac{x}{x_0}\right)^j - \left(\frac{x}{x_0}\right)^{p-1}\right]$$
$$+ (-1)^n f^{(pn)}(x_0)\left(\frac{x}{x_0}\right)^{p-1} + R_n(x, x_0, f), \quad (2.16)$$

where

$$R_n(x, x_0, f) = \int_0^{x_0} (-1)^n K_1\left(\frac{x}{x_0}, t\right) f^{(pn+p)}(t)\, dt \geqslant 0$$

with $K_1(x, t)$ given by (2.3). Therefore, from Lemma 2.2

$$(-1)^n f^{(pn)}(x) \geqslant M_{pn} \left(\frac{x}{x_0}\right)^{p-1} \geqslant M_{pn} x^{p-1} \geqslant M_{pn}(x^{p-1} - x^p). \quad (2.17)$$

For the interval $x_0 \leqslant x \leqslant 1$, we expand $f(x)$ using $f(1)$, $f(x_0)$ and $f^{(j)}(0)$ $(j = 1, 2, \ldots, p-2)$ to obtain

$$f(x) = f(x_0)\, D_0(x) + f(1)\, D_1(x) + \sum_{j=1}^{p-2} f^{(j)}(0)\, E_j(x) + R^*(x, x_0, f) \quad (2.18)$$

with

$$D_0(x) = \frac{1 - x^{p-1}}{1 - x_0^{p-1}}; \qquad D_1(x) = \left(\frac{1 - x_0^j}{1 - x_0^{p-1}}\right)(1 - x^{p-1})$$
$$j!E_j(x) = x^j - 1 + \left(\frac{1 - x_0^j}{1 - x_0^{p-1}}\right)(1 - x^{p-1}) \quad (2.19)$$

and, for $x_0 \leqslant x \leqslant 1$, $D_0(x) \geqslant 0$, $D_1(x) \geqslant 0$, $E_j(x) \geqslant 0$ $(j = 1, 2, \ldots, p-2)$. Furthermore [3, Lemma 7.3], $R^*(x, x_0, f) \geqslant 0$, $x_0 \leqslant x \leqslant 1$; so we have

$$(-1)^n f^{(pn)}(x) \geqslant M_{pn} \left(\frac{1 - x^{p-1}}{1 - x_0^{p-1}}\right) \geqslant M_{pn}(1 - x^{p-1}) \geqslant M_{pn}(x^{p-1} - x^p). \quad (2.20)$$

This proves the Lemma.

LEMMA 2.5. *Let $K_n(x, t)$ be defined by* (2.3) *and* (2.6). *If*

$$g_n(x) = \int_0^1 K_n(x, t)(t^{p-1} - t^p)\, dt \qquad (n = 1, 2,...) \tag{2.21}$$

then there exists a positive constant B such that

$$(-1)^n g_n(x_0) \geqslant \frac{B}{\lambda_1^{pn+p}}. \tag{2.22}$$

Proof. From (2.4)–(2.6), it follows easily that $g_n(x)$ has the properties

$$\begin{aligned} &g_n^{(pn)}(x) = x^{p-1} - x^p \\ &g_n^{(pk)}(1) = 0;\ g_n^{(pk+j)}(0) = 0 \qquad (j = 0, 1,..., p-2;\ k = 0, 1,..., n-1). \end{aligned} \tag{2.23}$$

Therefore,

$$g_n(x) = \frac{(p-1)!\, x^{pn+p-1}}{(pn+p-1)!} - \frac{p!\, x^{pn+p}}{(pn+p)!} + Q_{pn-1}(x), \tag{2.24}$$

where $Q_{pn-1}(x)$ is a polynomial of degree $pn - 1$. Thus by Lemma 2.1

$$g_n(x) = \sum_{k=0}^{n+1} \left[g_n^{(pk)}(1)\, C_{pk}(x) + \sum_{j=0}^{p-2} g_n^{(pk+j)}(0)\, A_{pk+j}(0) \right]. \tag{2.25}$$

By (2.23) and the properties of $C_{pk}(x), A_{pk+j}(x)$ ($j = 0, 1,..., p-2$; $k = 0, 1,...$), (2.25) yields

$$g_n(x) = \sum_{k=n}^{n+1} \left[g_n^{(pk)}(1)\, C_{pk}(x) + \sum_{j=0}^{p-2} g_n^{(pk+j)}(0)\, A_{pk+j}(x) \right]. \tag{2.26}$$

Using (2.24) and (2.25), (2.26) reduces to

$$g_n(x) = -p!\, (A_{pn+p}(x) + C_{pn+p}(x)]. \tag{2.27}$$

Therefore $(-1)^n g_n(x) = (-1)^{n+1} p!\, [A_{pn+p}(x) + C_{pn+p}(x)]$.
From (2.13) and (2.14) we have (2.22) which proves the lemma.

Proof of Theorem 2.1. From Lemma 2.1,

$$f(x) - \sum_{k=0}^{n-1} f^{(pk)}(1)\, C_{pk}(x) + \sum_{j=0}^{p-2} f^{(pk+j)}(0)\, A_{pk+j}(x)$$
$$= \int_0^1 K_n(x, t) f^{(pn)}(t)\, dt, \tag{2.28}$$

where $K_n(x, t)$ is defined by (2.3) and (2.6). Since $(-1)^n K_n(x, t) \geqslant 0$ $(0 \leqslant x \leqslant 1; 0 \leqslant t \leqslant 1, n = 1, 2,...)$ [3, Lemma 4.2] and by Lemma 2.4

$$\begin{aligned}\int_0^1 & K_n(x, t) f^{(pn)}(t)\, dt \\ &= \int_0^1 (-1)^n K_n(x, t)(-1)^n f^{(pn)}(t)\, dt \\ &\geqslant M_{pn} \int_0^1 (-1)^n K_n(x, t)(t^{p-1} - t^p)\, dt = M_{pn}(-1)^n g_n(x).\end{aligned}$$

Setting $x = x_0 (0 < x_0 < 1)$ and applying Lemma 2.3 we have

$$\int_0^1 K_n(x_0, t) f^{(pn)}(t)\, dt \geqslant \frac{BM_{pn}}{\lambda_1^{pn+p}} > 0. \tag{2.29}$$

From (2.7) we see that if the (p, L) series of $f(x)$ converges to $f(x)$ for $x = x_0(0 < x_0 < 1)$, the left side of (2.29) approaches zero, hence $M_{pn} \leqslant \epsilon_n \lambda_1^{pn}$ where $\epsilon_n \to 0$ as $n \to \infty$. It follows easily from a result of Hadamard (see, e.g., [2, p. 13]) that $M_{pn+j} \leqslant \epsilon_n \lambda_1^{pn+j}$ $(j = 0, 1,..., p-1)$. Therefore $f^{(k)}(x) = 0(\lambda_1{}^k)$ as $k \to \infty$, so if z is any complex number, we have

$$|f(z)| = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)\, z^n}{n!} \leqslant \sum_{n=0}^{\infty} \frac{M_n \mid z \mid^n}{n!} = 0(e^{|z|\lambda_1}) \text{ as } |z| \to \infty.$$

This proves the theorem.

THEOREM 2.2. *No condition of the form*

$$f(z) = 0\{\phi(|z|)\, e^{|z|\lambda_1}\} \qquad (|z| \to \infty) \tag{2.30}$$

where $\phi(r) \to 0$ as $r \to \infty$ and ϕ is independent of f is necessary for the absolute convergence of the (p, L) series to $f(z)$.

Proof. Choose a sequence of positive numbers $\{c_{pn}\}_{n=0}^{\infty}$ such that the series $\sum_{n=0}^{\infty} c_{pn}/\lambda_1^{pn}$ converges, where λ_1 is given by (1.4).

Define

$$f(z) = \sum_{n=0}^{\infty} (-1)^n c_{pn} C_{pn}(z). \tag{2.31}$$

If we set

$$C_{pn,1}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{N_{p,p-1}(zt)}{t^{pn+1} N_{p,p-1}(t)}\, dt, \tag{2.32}$$

where $N_{p,p-1}(t)$ is given by (1.3) and Γ is the circle $|t| = \frac{1}{2}(\lambda_1 + \lambda_2)$. From Cauchy's Residue Theorem it is easily shown that

$$C_{pn,1}(z) - C_{pn}(z) = \frac{(-1)^n\, pM_{p,p-1}(z\lambda_1)}{\lambda_1^{pn+1} M_{p,p-2}(\lambda_1)}. \tag{2.33}$$

Since $M_{p,p-1}(z\tau)$ is an entire function of exponential type $|\tau|$, and since $N_{p,p-1}(t)$ is uniformly bounded away from zero for $|t| = \frac{1}{2}(\lambda_1 + \lambda_2)$, we obtain the estimate

$$|C_{pn}(z)| \leqslant \left(\frac{2}{\lambda_1 + \lambda_2}\right) A \exp\left[\frac{1}{2}(\lambda_1 + \lambda_2)|z|\right]. \tag{2.34}$$

Therefore the series (2.31) is absolutely convergent for every z. Set $z = x + iy$ and $\omega = e^{2\pi i/p}$. In order to complete the proof of Theorem 2.2, we require a lower bound on $|f(\sqrt{\omega} y)|$, and so we prove

LEMMA 2.6. *For $n = 0, 1,\ldots$ and $y \geqslant 0$,*

$$-\sqrt{\omega}(-1)^n\, C_{pn}(\sqrt{\omega} y) \geqslant pA_p \sum_{j=0}^{n-1} \frac{y^{pk+p-1}(\lambda_1)^{p(n-j-1)+2}}{(pk + p - 1)!}, \tag{2.35}$$

where A_p is a positive constant depending only on p.

Proof of Lemma 2.6. Since we have [3, p. 15]

$$C_{pn}(x) = p(-1)^{n+1} \sum_{k=1}^{\infty} \frac{M_{p,p-1}(x\lambda_k)}{M_{p,p-2}(\lambda_k)\, \lambda_k^{pn+1}}, \tag{2.36}$$

where $\lambda_k (k = 1, 2,\ldots)$ is given by (1.4). This representation of $C_{pn}(x)$ is valid in [0, 1].

Case I: $p = 2$. Then (2.36) reduces to the Fourier Series expansion of $C_{2n}(x)$ and we have $A_2 = \frac{3}{4}$, whereas Boas [1, p. 242] obtains the better estimate $\frac{3}{2}$ by summing a geometric series.

Case II: $p > 2$. Boas' technique does not work, since the function $M_{p,p-1}(t)$ is not periodic for $p > 2$. However, using the properties of the function $M_{p,p-1}(t)$ [4, p. 46] we have

$$C_{pn}^{(pj+\nu)}(0) = 0 \qquad (\nu = 0, 1,\ldots, p - 2) \tag{2.37}$$

$$\begin{aligned} C_{pn}^{(pj+p-1)}(0) &= p(-1)^{n+j+1} \sum_{k=1}^{\infty} \frac{1}{M_{p,p-2}(\lambda_k)\, \lambda_k)^{p(n-j-1)+2}} \\ &= \frac{p(-1)^{n+j}}{(\lambda_1)^{p(n-j-1)+2}} \sum_{k=1}^{\infty} \left(-\frac{1}{M_{p,p-2}(\lambda_k)}\right)\left(\frac{\lambda_1}{\lambda_k}\right)^{p(n-j-1)+2} \\ &= \frac{p(-1)^{n+j} A_{p,j}}{(\lambda_1)^{p(n-j-1)+2}}. \end{aligned} \tag{2.38}$$

Now for $p > 2$, $\{(-1)^k M_{p,p-2}(\lambda_k)\}_{k=1}^{\infty}$ is an increasing sequence of positive numbers unbounded above [4], so for $j = 0, 1,..., n-1$, we have

$$A_{p,j} \geqslant -\frac{1}{M_{p,p-2}(\lambda_1)} + \frac{1}{M_{p,p-2}(\lambda_2)}\left(\frac{\lambda_1}{\lambda_2}\right)^{p(n-j-1)+2}$$

$$> -\frac{1}{M_{p,p-2}(\lambda_1)}\left[1-\left(\frac{\lambda_1}{\lambda_2}\right)^2\right] = A_p > 0.$$

From (2.38) and (2.39) we have

$$(-1)^{n+j}\, C_{pn}^{(pj+p-1)}(0) > \frac{pA_p}{(\lambda_1)^{p(n-j+1)+2}}\ (j = 0, 1,..., n-1). \qquad (2.40)$$

Since $C_{pn}^{(pn+p-1)}(x) = C_0^{(p-1)}(x) = (p-1)!$, we use the Maclaurin series to express $C_{pn}(\sqrt{\omega}y)$ in the form

$$(-1)^n D_{pn}(\omega y) = (-1)^n \sum_{j=0}^{pn+p-1} \frac{(\sqrt{\omega}y)^j}{j!} C_{pn}^{(j)}(0)$$

$$= (-1)^n \sum_{k=0}^{n} \frac{(\sqrt{\omega}y)^{pk+p-1}}{(pk+p-1)!} C_{pn}^{(pk+p-1)}(0)$$

$$= -\frac{1}{\sqrt{\omega}} \sum_{k=0}^{n} \frac{(-1)^{k+n}\, y^{pk+p-1}}{(pk+p-1)!} C_{pn}^{(pk+p-1)}(0).$$

Therefore, we have

$$-\sqrt{\omega}(-1)^n C_{pn}(\sqrt{\omega}y) \geqslant pA_p \sum_{k=0}^{n-1} \frac{y^{pk+p-1}}{(pk+p-1)!} (\lambda_1)^{p(n-k-1)+2}. \qquad (2.41)$$

This completes the proof of Lemma 2.6.
Substituting (2.41) in (2.31), we have, for $y \geqslant 0$,

$$-\sqrt{\omega}\, f(\sqrt{\omega}y) = -\sqrt{\omega} \sum_{n=0}^{\infty} (-1)^n c_{pn} C_{pn}(\sqrt{\omega}y)$$

$$\geqslant \frac{pA_p}{\lambda_1} \sum_{n=1}^{\infty} c_{pn}(\lambda_1)^{-pn} \sum_{k=0}^{n-1} \frac{(y\lambda_1)^{pk+p-1}}{(pk+p-1)!}$$

$$= \frac{pA_p}{\lambda_1} \sum_{k=1}^{\infty} \frac{q_k(y\lambda_1)^{pk+p-1}}{(pk+p-1)!}$$

where

$$q_k = \sum_{n=k+1}^{\infty} c_{pn}(\lambda_1)^{-pn} \quad \text{and} \quad q_k \to 0 \quad \text{as} \quad k \to \infty.$$

Given any number $c > 0$, when $cy = p - 1 \pmod{p}$ we have, using Stirling's formula

$$\begin{aligned}
&\frac{-\sqrt{\omega}\,\lambda_1}{pA_p} f(\sqrt{\omega} y) \\
&\quad \geqslant q_{cy} \sum_{k=0}^{cy} \frac{(y\lambda_1)^{pk+p-1}}{(pk+p-1)!} \\
&\quad = q_{cy} \left\{ N_{p,p-1}(y\lambda_1) - \sum_{n=cy+1}^{\infty} \frac{(y\lambda_1)^{pk+p-1}}{(pk+p-1)!} \right\} \\
&\quad = q_{cy} \left\{ N_{p,p-1}(y\lambda_1) - \frac{(y\lambda_1)^{cy+1}}{(cy+1)!} N_{p,p-1}(y\theta\lambda_1) \right\}, \qquad (0 < \theta < 1) \\
&\quad \geqslant q_{cy} \left\{ N_{p,p-1}(y\lambda_1) \left[1 - \left(\frac{e\lambda_1}{c} \right)^{cy+1} \right] \right\}.
\end{aligned}$$

Now if c is chosen so large that $\lambda_1/c < 1/e$, then we have

$$\frac{-\sqrt{\omega}\,\lambda_1}{pA_p} f(\sqrt{\omega} y) \geqslant [1 - o(1)]\, q_{cy} N_{p,p-1}(y\lambda_1).$$

Therefore, for any given function $\phi(r)$ such that $\phi(r) \to 0$ as $r \to \infty$, we may choose the sequence $\{c_{pn}\}_{n=0}^{\infty}$ so that q_{cy} will approach zero as slowly as desired, so there is a function $f(z)$ defined by the (p, L) series (2.31) not satisfying (2.30). This completes the proof of Theorem 2.2.

3. Some Sufficient Conditions for Representation of Functions by (p, L) Series

The results of this section generalize the results given by Boas [1] for the Lidstone series. Here we obtain similar results for (p, L) series.

Theorem 3.1. *The (p, L) series of $f(z)$ converges to $f(z)$ if*

$$f(z) = 0(|z|^{-1/2} e^{|z|\lambda_1}) \qquad (|z| \to \infty) \tag{3.1}$$

where λ_1 is given by (1.4).

Before proving Theorem 3.1, we require

LEMMA 3.1. *Let* $K_n(x, t)$ $(n = 1, 2, \ldots,)$ *be defined by* (2.2) *and* (2.5). *Then for* $0 \leqslant x \leqslant 1$

$$0 \leqslant (-1)^n \int_0^1 K_n(x, t)\, dt \leqslant \frac{B}{\lambda_1^{pn}} \tag{3.2}$$

The proof of Lemma 3.1 is given [3, Lemma 6.4, p. 18].

Proof of Theorem 3.1. If $f(z)$ satisfies (3.1) then

$$|f(z)| \leqslant \eta(|z|)|z|^{-1/2} e^{|z|\lambda_1} \tag{3.3}$$

where $\eta(s) \to 0$ as $s \to \infty$. Furthermore, for $0 \leqslant t \leqslant 1$ we have

$$f^{(pn+j)}(t) = \frac{(pn+j)!}{2\pi i} \int_{C_j} \frac{f(z)\, dz}{(z-t)^{pn+j+1}} \qquad (j = 0, 1, \ldots, p-2), \tag{3.4}$$

where C_j is the circle $|z| = s^{(j)} > 1$, and

$$s^{(j)} = 1 + \frac{pn+j}{\lambda_1}. \tag{3.5}$$

From (3.3) and (3.4) we have

$$\begin{aligned} |f^{(pn+j)}(t)| &\leqslant \frac{(pn+j)!\, s^{(j)}}{(s^{(j)}-1)^{pn+j+1}} \left(\max_{|z| \leqslant s^{(j)}} |f(z)| \right) \\ &\leqslant \frac{(pn+j)!\, \eta(s^{(j)})\, e^{\lambda_1 s^{(j)}}}{(s^{(j)}-1)^{pn+j+1} (s^{(j)})^{1/2}}. \end{aligned} \tag{3.6}$$

From (3.5) and Stirling's formula, we have

$$\begin{aligned} |f^{(pn+j)}(t)| &\leqslant \frac{B(pn+j)^{pn+j} e^{-pn-j} \eta^{1/2} (s^{(j)})^{1/2}\, \eta(s^{(j)})\, e^{\lambda_1 s^{(j)}}}{(s^{(j)}-1)^{pn+j+1}} \\ &\leqslant B(\lambda_1)^{pn}\, \delta_j(n) \qquad (0 \leqslant t \leqslant 1), \end{aligned}$$

where

$$\delta_j(n) = \frac{\lambda_1}{p}\, \eta(s^{(j)}). \tag{3.8}$$

Let $S_N(x)$ be the sum of the first N terms of the (p, L) series (1.5). Then, using (3.7), Lemmas 2.1, 2.2 and 3.1, we have

$$|f(x) - S_{pn}(x)| = \left| \int_0^1 K_n(x, t) f^{(pn)}(t)\, dt \right| \leqslant B\delta_0(n) \to 0 \tag{3.9}$$

as $n \to \infty$. Furthermore, using (3.6) and inequalities (2.11) and (2.12) we have

$$| S_{pn+k}(x) - S_{pn}(x)| = \left| f^{(pn)}(1)\, C_{pn}(x) + \sum_{j=0}^{k} f^{(pn+j)}(0)\, A_{pn+j}(x) \right|$$

$$\leqslant B(\lambda_1)^{pn}\, \delta_0(n) \left\{ | C_{pn}(x)| + \sum_{j=0}^{k} | A_{pn+j}(x)| \right\} \leqslant B\delta_0(n);$$

so we have

$$|f(x) - S_{pn+k}(x)| \to 0 \text{ as } n \to \infty \ (k = 0, 1,..., p - 1)$$

and this proves Theorem 3.1.

THEOREM 3.2. *The (p, L) series converges absolutely to $f(z)$ provided*

$$f(z) = 0\{\eta(| z |)| z |^{-1/2} e^{|z|\lambda_1}\} \qquad (| z | \to \infty) \tag{3.10}$$

where $\eta(r) \downarrow 0$ and $\int^{\infty} \eta(r)\, dr$ converges.

Proof. From (2.34) we easily obtain the estimate

$$| C_{pn}(z)| \leqslant \frac{Be^{|z|\lambda_2}}{(\lambda_1)^{pn}}. \tag{3.11}$$

A similar procedure yields the estimate

$$| A_{pn+j}(z)| \leqslant \frac{Be^{|z|\lambda_2}}{(\lambda_1)^{pn}}, \tag{3.12}$$

where (3.11) and (3.12) are valid for all complex z with B a suitable constant and λ_1 and λ_2 are given by (1.4).

If we set $t = 1$ for $j = 0$, and $t = 0$ for $j = 0, 1,..., p - 2$ in (3.7), then using (3.11) and (3.12), the (p, L) series of $f(z)$ (given by (1.5)) is dominated by the series $\sum_{j=0}^{p-2} AB\delta_j(n)$ where A and B are suitable constants. Therefore (1.5) converges absolutely provided $\sum_{n=0}^{\infty} \delta_j(n)$ converges for $j = 0, 1,..., p - 2$. From (3.8) and since $\int^{\infty} \eta(r)\, dr$ converges, Theorem 3.2 is proved.

THEOREM 3.3. *The (p, L) series may fail to converge when*

$$f(z) = 0(e^{|z|\lambda_1}) \qquad (| z | \to \infty) \tag{3.13}$$

Proof. After Boas [1], consider the function $f(z) = e^{z\lambda_1}$ which satisfies (3.13). However, from (2.41)

$$- \sqrt{\omega}(-1)^n\, C_{pn}(\sqrt{\omega}y) \geqslant \frac{pA_p\, y^{p-1}}{(\lambda_1)^{pn+p-2}} \qquad (y \geqslant 0). \tag{3.14}$$

Now $f^{(pn)}(1) = \lambda_1^{pn} e^{\lambda_1}$, so the terms of the form $f^{(pn)}(1)\, C_{pn}(z)$ $(n = 0, 1,...)$ of (1.5) do not approach zero when $z = \sqrt{\omega} y$ $(\omega = e^{2\pi i/p})$, hence (1.5) cannot converge.

THEOREM 3.4. *The (p, L) series may fail to converge absolutely when*

$$f(z) = 0(|z|^{-1} e^{|z|\lambda_1}) \qquad (|z| \to \infty) \tag{3.15}$$

where λ_1 is given by (1.4).

Proof. After Boas [1], we consider the function

$$f(z) = \frac{e^{\lambda_1(z-1)} - 1}{\lambda_1(z-1)} = 1 + \sum_{n=2}^{\infty} \frac{[\lambda_1(z-1)]^{n-1}}{n!}$$

Now

$$f^{(pn)}(1) = \frac{\lambda_1^{pn}}{pn+1}$$

so from (2.13) we have for fixed $x_0 (0 < x_0 < 1)$

$$(-1)^n C_{pn}(x_0) \geqslant \frac{B}{\lambda_1^{pn}} \qquad (n = 1, 2,...).$$

Therefore, the (p, L) series of $f(z)$ cannot converge absolutely for $z = x (0 < x < 1)$. However, by Theorem 3.1, the function $f(z)$ is represented by its (p, L) series (1.5).

ACKNOWLEDGMENT

I wish to express my thanks to Professor A. Sharma, University of Alberta, Edmonton, Canada, for several useful discussions and suggestions during the preparation of this paper.

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